On a nonlinear nonlocal ODE arising in magnetic recording

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Abstract

In this paper we study the uniqueness of the solution for a nonlinear ODE with nonlocal terms. We consider a limit case of a one-dimensional equation arising in magnetic recording. The equation models the tape deflection where the magnetic head profile, with trenches to control the tape position, is a known function.

Keywords: Nonlocal terms; Lubrication theory; Reynolds equation; Nonlinear elliptic equations

1. Introduction

Different kinds of nonlocal terms appear in a great number of partial differential equations of elliptic type. In this work we will consider a particular case where the unknown u appears evaluated at a distinguished point x_0 of the domain. The simplest example of an elliptic problem with this type of term is the following:

$$
\begin{aligned}
&-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \lambda u\left(\frac{1}{2}\right), \qquad x \in (0,1), \lambda > 0,\\
&u(0) = u(L) = 0.
\end{aligned}
$$

The solution of the problem depends on the value of lambda: for $\lambda \neq 8$ the only solution is $u = 0$, whereas for $\lambda = 8$ infinitely many solutions appear. These are given by $u = cx(x-1)$ for any $c \neq 0$. Notice that this is not an eigenvalue problem and therefore the question of uniqueness is significant.

In the next section we present a problem arising in magnetic recording that we will study in Section 3.

2. The magnetic tape

A magnetic tape is driven with constant velocity over the magnetic head and its position u is given as the solution of the ODE

$$
\begin{cases}
-\dfrac{\partial^2 u}{\partial x^2} = k\left(\dfrac{u(L_1)-\delta(L_1)}{u(x)-\delta(x)} - 1\right)\chi_{[L_1,L_2]} & 0 < x < L,\\
u(0) = u(L) = 0
\end{cases}
\tag{2.1}
$$

where $0 < L_1 < L_2 < L$, $\chi_{[L_1,L_2]}$ is the characteristic function of the interval $[L_1, L_2]$, δ is the head profile, k is a positive constant and u satisfies

$$
u(x) > \delta(x) \qquad \text{if } L_1 \le x \le L_2. \tag{2.2}
$$

(2.1) is the limit case of a system where the pressure p of the air is modelled by the compressible Reynolds equation and the position of the tape u is modelled by the beam equation (see [1–3]). Problem (2.1) has been analyzed in [4] and [5].

In [4] existence and uniqueness is proved using a shooting method under the assumption

$$
\delta \in C^2 \text{ and } \delta''(x) < 0, \qquad L_1 \le x \le L_2. \tag{2.3}
$$

This assumption is very restrictive, mathematically and physically, because magnetic heads do not usually satisfy the concavity condition (2.3) and are generally discontinuous (see [2,3,5]).

In [5] the existence of solutions is proved using a sub- and super-solution method under more general assumptions:

$$
\begin{aligned}
&\delta \text{ is piecewise continuous with jump discontinuous at } \xi_1, \ldots, \xi_s \text{ where}\\
&\xi_0 = L_1 < \xi_1 \ldots \xi_s < L_2 = \xi_{s+1}, \qquad \text{and } \delta \in C^1[\xi_i, \xi_{i+1}] \text{ for } 0 \le i \le s,
\end{aligned}
\tag{2.4}
$$

and

$$
\delta(L_1) < \delta'(L_1)L_1, \qquad \delta(L_2) < (L_2 - L)\delta'(L_2). \tag{2.5}
$$

Uniqueness was proved in case (2.3), but not for the general case (2.4), (2.5). The question of uniqueness is not just a mere mathematical issue. Its analysis is also necessary for simulating the solution with a numerical approach.

The main result of this paper is enclosed in the following theorem.

Theorem 2.1. *Assume that* (2.4) *and* (2.5) *are satisfied. Then there exists a unique solution u to* (2.1) *satisfying* (2.2).

Note that the inequality $\delta(L_1) < \delta'(L_1)L_1$ means that the tangent to the head at $x = L_1$ intersects the x-axis in the interval $(0, L_1)$. Similarly, the second inequality in (2.5) means that the tangent to the head at $x = L_2$ intersects the x-axis in the interval (L_2, L).

3. Proof of the Theorem 2.1

By [5, Theorem 2.1] we know that any solution u to (2.1) satisfies

$$u \in W^{2,\infty}(0, L). \tag{3.1}$$

We assume without loss of generality that

$$\delta(L_1) = 0, \qquad \delta(x) \geq 0, \qquad \delta \geq \frac{k}{2}(x - L_1)^2 \text{ if } x \in [L_1, L_2]. \tag{3.2}$$

Remark 3.1. Notice that if δ does not satisfy (3.2) we can introduce the change

$$\tilde{u} = u - \delta(L_1) + \gamma(x - L_1), \qquad \tilde{\delta}(x) = \delta - \delta(L_1) + \gamma(x - L_1)$$

where γ, defined by

$$\gamma = \max\left\{0, -\min_{x\in(L_1,L_2)}\left\{\frac{\delta(x) - \delta(L_1)}{x - L_1}\right\}\right\} + k(L_2 - L_1)$$

is bounded by (2.4). Then $\tilde{\delta}$ satisfies (3.2) and $\tilde{u}$ satisfies

$$\begin{cases} -\dfrac{\partial^2 \tilde{u}}{\partial x^2} = k\left(\dfrac{\tilde{u}(L_1)}{\tilde{u}(x) - \tilde{\delta}(x)} - 1\right)\chi_{[L_1,L_2]}, & 0 < x < L, \\ \tilde{u}(0) = -\delta(L_1) - \gamma L_1, & \tilde{u}(L) = -\delta(L_1) + \gamma(L - L_1), \\ \tilde{u}(x) - \tilde{\delta}(x) > 0, & \text{if } L_1 \leq x \leq L_2. \end{cases} \tag{3.3}$$

As in [5], we consider the unique solution $u(\lambda)$ of the problem

$$\begin{cases} -\dfrac{\partial^2}{\partial x^2}u(\lambda) = k\left(\dfrac{\lambda}{u(\lambda) - \delta(x)} - 1\right)\chi_{[L_1,L_2]}, & 0 < x < L, \\ u(\lambda, 0) = \gamma_1, & u(\lambda, L) = \gamma_2, \\ u(x) - \delta(x) > 0 & \text{if } L_1 \leq x \leq L_2, \end{cases} \tag{3.4}$$

for $\gamma_1 > -L_1\delta'(L_1)$ and $\gamma_2 > (L - L_2)\delta'(L_2) + \delta(L_2)$. By [5, Lemma 2.1] we know that for any $\lambda > \delta(L_1) = 0$ there exists a unique solution $u(\lambda) > \delta$ to (3.4).

Lemma 3.1. *If* $\lambda_1 > \lambda_2$ *then* $u(\lambda_1) \geq u(\lambda_2)$ *in* $[0, L]$.

Proof. Consider $u(\lambda_1) - u(\lambda_2)$ which satisfies

$$\begin{aligned} &-\frac{\partial^2}{\partial x^2}(u(\lambda_1) - u(\lambda_2)) - \left(\frac{\lambda_1}{u(\lambda_1) - \delta(x)} - \frac{\lambda_1}{u(\lambda_2) - \delta(x)}\right)\chi_{[L_1,L_2]} \\ &\quad = (\lambda_1 - \lambda_2)\frac{1}{u(\lambda_2) - \delta(x)}\chi_{[L_1,L_2]} \geq 0. \end{aligned} \tag{3.5}$$

Let us consider the continuous and Lipschitz function ϕ defined by $\phi(s) = s$ if $s < 0$ and 0 otherwise. Let us take $\phi(u(\lambda_1) - u(\lambda_2))$ as a test function in (3.5); we obtain

$$\int_0^L [(u_x(\lambda_1) - u_x(\lambda_2))]^2 \phi'(u(\lambda_1) - u(\lambda_2))\mathrm{d}x + \int_{L_1}^{L_2} \left(\frac{\lambda_1}{u(\lambda_1) - \delta} - \frac{\lambda_1}{u(\lambda_2) - \delta}\right) \times \phi(u(\lambda_1) - u(\lambda_2))\mathrm{d}x = (\lambda_1 - \lambda_2) \int_{L_1}^{L_2} \frac{1}{u(\lambda_2) - \delta}\phi(u(\lambda_1) - u(\lambda_2))\mathrm{d}x \leq 0.$$

Since $\frac{1}{u-\delta}$ is decreasing (as a function of u) for $u > \delta$, we obtain

$$\left(\frac{\lambda_1}{u(\lambda_1) - \delta} - \frac{\lambda_1}{u(\lambda_2) - \delta}\right) \phi(u(\lambda_1) - u(\lambda_2)) \leq 0$$

and then

$$\int_0^L [(u_x(\lambda_1, x) - u_x(\lambda_2, x))]^2 \phi'(u(\lambda_1) - u(\lambda_2))\mathrm{d}x \leq 0.$$

By definition of ϕ we deduce the desired result. □

Let us argue by contradiction and consider that there exist two different solutions, u_1 and u_2, to (3.3) such that $u_1(L_1) = \lambda_1$, $u_2(L_1) = \lambda_2$ and

$$\lambda_1 > \lambda_2. \tag{3.6}$$

Then, u_i (for $i = 1, 2$) satisfies

$$-\frac{\partial^2 u_i}{\partial x^2} = k\left(\frac{\lambda_i}{u_i(x) - \delta(x)} - 1\right)\chi_{[L_1, L_2]} \qquad 0 < x < L, \tag{3.7}$$

$$u_i(0) = -\delta(L_1) - \gamma L_1, \qquad u_i(L) = -\delta(L_1) + \gamma(L - L_1). \tag{3.8}$$

Consider the new unknown v and w defined by

$$v = u_1 - u_2, \qquad w = \lambda_2 u_1 - \lambda_1 u_2 \qquad \text{in } [L_1, L_2].$$

Then v satisfies

$$-\frac{\partial^2 v}{\partial x^2} = k\left(\frac{\lambda_1}{u_1(x) - \delta(x)} - \frac{\lambda_2}{u_2(x) - \delta(x)}\right) \qquad L_1 < x < L_2, \tag{3.9}$$

$$v(L_1) = \lambda_1 - \lambda_2, \qquad v_x(L_1) = \frac{\lambda_1 - \lambda_2}{L_1} \tag{3.10}$$

and w satisfies

$$-\frac{\partial^2 w}{\partial x^2} = k\left(\frac{\lambda_2\lambda_1}{u_1(x) - \delta(x)} - \frac{\lambda_1\lambda_2}{u_2(x) - \delta(x)} - (\lambda_2 - \lambda_1)\right) \qquad 0 < x < L, \tag{3.11}$$

$$w(L_1) = w_x(L_1) = 0. \tag{3.12}$$

Since

$$\frac{\lambda_1}{u_1(x) - \delta(x)} - \frac{\lambda_2}{u_2(x) - \delta(x)} = \frac{\lambda_1 u_2(x) - \lambda_2 u_1(x) - (\lambda_1 - \lambda_2)\delta(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))}$$
$$= \frac{-w - (\lambda_1 - \lambda_2)\delta(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))}$$

we obtain by (3.6) and (3.2) that $-(\lambda_1 - \lambda_2)\delta(x) \le 0$. Then, writing

$$f(x) = \frac{1}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))} > 0$$

we obtain

$$v_{xx} = kf(x)(w + (\lambda_1 - \lambda_2)\delta), \qquad x \in (L_1, L_2). \tag{3.13}$$

In the same way,

$$\frac{1}{u_1(x) - \delta(x)} - \frac{1}{u_2(x) - \delta(x)} = \frac{u_2(x) - u_1(x)}{(u_1(x) - \delta(x))(u_2(x) - \delta(x))} = -f(x)v$$

and then

$$w_{xx} = k\lambda_1\lambda_2 f v - k(\lambda_1 - \lambda_2), \qquad x \in (L_1, L_2). \tag{3.14}$$

Lemma 3.2. $w \ge -\frac{k}{2}(\lambda_1 - \lambda_2)(x - L_1)^2$ *in* $[L_1, L_2]$.

By Lemma 3.1 we deduce that $v \ge 0$ and by (3.14) we get

$$w_{xx} \ge -k(\lambda_1 - \lambda_2) \qquad \text{if } (L_1, L_2). \tag{3.15}$$

Integrating (3.15) twice over (L_1, x), as a result of (3.12) we obtain the desired result. □

End of the Proof of the Theorem. By the previous lemma and from (3.13) we deduce

$$w + (\lambda_1 - \lambda_2)\delta \ge (\lambda_1 - \lambda_2)\left(-\frac{k}{2}(x - L_1)^2 + \delta\right).$$

By (3.2) it results that $(\lambda_1 - \lambda_2)(-\frac{k}{2}(x - L_1)^2 + \delta) \ge 0$ and substituting this in (3.13) we get

$$v_{xx} \ge 0 \qquad \text{in } (L_1, L_2). \tag{3.16}$$

Since $v(L_1) > 0$, $v_x(L_1) > 0$ (see (3.10)) and from (3.16), $v(L_2)$ satisfies

$$0 < v(L_2) = u_1(L_2) - u_2(L_2) \tag{3.17}$$

and $v_x(L_2)$

$$0 < v_x(L_2) = u_{1x}(L_2) - u_{2x}(L_2). \tag{3.18}$$

Integrating (3.7) in the interval (L_2, L) we obtain

$$u_1(L) = u_1(L_2) + (L - L_2)u_{1x}(L_2),$$

and

$$u_2(L) = u_2(L_2) + (L - L_2)u_{2x}(L_2).$$

Subtracting the above expressions we get

$$u_1(L) - u_2(L) = u_1(L_2) - u_2(L_2) + (L - L_2)(u_{1x}(L_2) - u_{2x}(L_2))$$

and by (3.17) and (3.18) it results that $u_1(L) - u_2(L) > 0$ which contradicts (3.8). □

Remark 3.2. The typical head profile satisfies

$$\delta(x) - \delta(L_1) \le \delta'(L_1)(x - L_1) \qquad \text{in } [L_1, L_2]. \tag{3.19}$$

Then (2.5) is a necessary assumption.

If $\delta'(L_1) \leq \frac{\delta(L_1)}{L_1}$ and δ satisfies (3.19), then $u'(L_1) \geq \frac{\delta(L_1)}{L_1} > 0$ and $\frac{u(L_1)-\delta(L_1)}{u(x)-\delta(x)}$ is decreasing (as a function of x). We obtain $u(L_2) > u(L_1) > 0$ and $u_x(L_2) > u_x(L_1) > 0$ and then $u(L) > 0$, which contradicts (2.1).

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